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FAKE ENRIQUES SURFACES

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0. INTRODUCTION

AN ENRIQUES surface X is a projective algebraic surface of Kodaira dimension 0 with fundamental group $\pi_1(X) = \mathbb{Z}/2$. A homotopy Enriques surface is a compact complex surface which is oriented homotopy equivalent to an "ordinary" Enriques surface.

The purpose of this paper is to study these surfaces in three different categories: as complex surfaces, as topological manifolds and as differentiable manifolds.

We obtain a fairly complete classification in the complex analytic category.

THEOREM. *The homotopy Enriques surfaces are precisely those regular elliptic surfaces with $p_g = 0$, which have two multiple fibres $2pF_{2p}$, $2qF_{2q}$, where p, q are relatively prime, odd, positive integers.*

We write $X_{2p,\,2q}$ for a surface of this type. The ordinary Enriques surfaces appear as the surfaces of type $X_{2,\,2}$ i.e. $p = q = 1$. All other $X_{2p,\,2q}$ are "fake" Enriques surfaces; they have Kodaira dimension 1.

The difficult part in the proof of this theorem is to show that the surfaces $X_{2p,\,2q}$ all have the same oriented homotopy type if $p,q \geqq 1$ are relatively prime and odd. We do this using a criterion of Hambleton and Kreck [15]. These authors show that the oriented homotopy type of a closed, oriented, topological 4-manifold X with $\pi_1 = \pi_1(X)$ finite with 4-periodic cohomology is determined by its quadratic 2-type. This quadratic 2-type is essentially given by the equivariant intersection form $S_{\tilde{X}}$ of the universal covering $\tilde{X}$ and the first k-invariant $k(X) \in H^3(\pi_1, \pi_2)$, where $\pi_2 = \pi_2(X)$ is considered as a π_1-module.

To compute the equivariant intersection form of the universal covering $\tilde{X}_{2p,\,2q}$ of a surface $X_{2p,\,2q}$, we prove that $\tilde{X}_{2p,\,2q}$ is a homotopy $K3$ surface in the sense of Kodaira [20]. Then we use the uniqueness of the embedding of the Enriques lattice multiplied by 2 into the $K3$ lattice [26] to identify the equivariant intersection form of $\tilde{X}_{2p,\,2q}$ the $K3$ lattice $2\langle -E_8\rangle \oplus 3\langle H\rangle$ with the standard $\mathbb{Z}/2$-action coming from the Enriques involution [2]. Given this equivariant intersection form there are essentially two possible k-invariants

$$k \in H^3(\mathbb{Z}/2;\, 2\langle -E_8\rangle \oplus 3\langle H\rangle) \cong H/2H,$$

namely (1, 0) and (1, 1). Both occur for topological manifolds. But (1, 0) can not be realized by a differentiable manifold. Thus all $X_{2p,\,2q}$ must have k-invariant (1, 1) and hence the same oriented homotopy type.

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In the next step we analyze the topological structures of homotopy Enriques surfaces.

THEOREM. *Every two homotopy Enriques surfaces are homeomorphic to each other.*

The proof uses topological surgery in dimension 4 [18]. According to Freedman [12] and Wall [31] there exists an exact surgery sequence

$$L_5^s(\pi_1, w_1) \to \mathscr{S}_{\mathrm{top}}(X) \xrightarrow{\nu} \mathscr{N}_{\mathrm{top}}(X) \xrightarrow{\vartheta} L_4^s(\pi_1, w_1)$$

for every closed topological 4-manifold X with "good" fundamental group $\pi_1 = \pi_1(X)$.

For an oriented manifold with finite cyclic fundamental group one can use this sequence to identify the set of topological smoothings $\mathscr{S}_{\mathrm{top}}(X)$ with the cohomology group $H^2(X; \mathbb{Z}/2)$.

The set of topological structures in the oriented simple homotopy type of X is the orbit space $\mathscr{S}_{\mathrm{top}}(X)/\mathrm{Aut}(X)$ of $\mathscr{S}_{\mathrm{top}}(X)$ under the natural action of the group $\mathrm{Aut}(X)$ of homotopy classes of simple self equivalences.

Using the vanishing of the Whitehead group $\mathrm{Wh}(\pi_1)$ for $\pi_1 = \mathbb{Z}/2$ and computations of Wall [31] we find that there can be at most two oriented, topological manifolds in the homotopy type of an Enriques surface. The last step in the proof of the theorem is due to Kreck [21]; he shows that there is in fact only one such manifold.

Now we can ask for the C^∞-classification of the homotopy Enriques surfaces $X_{2p,2q}$. Here we have the following partial answer:

THEOREM. *Fix an arbitrary pair (p_o, q_o) of relatively prime, odd, positive integers. There exist at most finitely many other such pairs (p, q) for which a surface of type $X_{2p,2q}$ can be diffeomorphic to a surface of type $X_{2p_o,2q_o}$.*

In the special case $p_o = p = 1$ we show:

COROLLARY. *If two homotopy Enriques surfaces of type $X_{2,2q}$ and $X_{2,2q_o}$ are diffeomorphic, then $q = q_o$.*

It is of course easy to construct homotopy Enriques surfaces of type $X_{2p,2q}$ for any given pair (p, q) (relatively prime, odd, positive), simply by applying two logarithmic transformations with multiplicities $2p$ and $2q$ at two non-singular fibres of a rational elliptic surface [2]. Therefore we have:

COROLLARY. *There exist infinitely many different smooth structures on the topological manifold underlying an Enriques surface.*

The main step in the proof of these results is a computation of Donaldson's Γ-invariants [9] for the surfaces $X_{2p,2q}$. This allows us to distinguish their differentiable structures. In general this invariant is a complicated function defined on the set of chambers in the positive cone of an oriented Riemannian 4-manifold with $b_1 = 0$ and $H^2_+ \cong \mathbb{R}$. Its definition is in terms of certain cohomology classes associated to a compactification of the moduli space $M(g)$ of $*_g$-anti-self-dual connections on the principal SU(2)-bundle P with $c_2(P) = 1$, g a Riemannian metric. If the base manifold is a projective surface with a Hodge metric corresponding to an ample line bundle H, the subspace of irreducible connections in $M(g)$ has an algebro-geometric description. It can be identified with the moduli space $M(H)$ of H-stable rank-2 vector bundles E with trivial determinant and $c_2(E) = 1$ [5], [22].

We describe the structure of these spaces for generic surfaces of type $X_{2p,2q}$.

There are two points which simplify things considerably in our situation: First of all, since the intersection forms of homotopy Enriques surfaces are even, there are no reducible connections on P. Secondly, since $\pi_1 = \mathbb{Z}/2$ the moduli spaces $M(H)$ are already complete, which allows us to compute the Donaldson invariants easily. In particular we find that a "fake" Enriques surface is also "fake" in the differentiable category. This consequence is in remarkable contrast to a recent result of Ue[29]. He shows that the oriented diffeomorphism type of a relatively minimal elliptic surface with positive Euler characteristic $e > 0$ and fundamental group π_1 is determined by these two invariants unless π_1 is cyclic. On the other hand, it can be shown that for any $k \geqq 1$ there are always infinitely many diffeomorphism types induced by regular elliptic surfaces with $e = 12$ and $\pi_1 = \mathbb{Z}/k$ [13], [23], [27].

1. FAKE ENRIQUES SURFACES

By a surface we shall mean a compact complex manifold of dimension 2. We denote by K a canonical divisor, by $p_g(X)$ and $q(X)$ the geometric genus and irregularity of a surface X.

An Enriques surface is an algebraic surface X with $q(X) = 0$, for which $2K \sim 0$ but $K \nsim 0$. Every Enriques surface X admits an elliptic fibration $\pi: X \to \mathbb{P}^1$ over the projective line. Such a fibration has exactly two multiple fibres $2F'$ and $2F''$; a canonical divisor is given by $K \sim F' - F''$. The fundamental group of X is $\mathbb{Z}/2$, the universal covering $\tilde{X}$ is a $K3$ surface. The first Chern class $c_1: \mathrm{Pic}(X) \to H^2(X; \mathbb{Z}) \cong \mathbb{Z}^{10} \oplus \mathbb{Z}/2$ is an isomorphism, $c_1(K)$ generates the torsion group. The intersection pairing on the free part $H^2(X; \mathbb{Z})_f = H^2(X; \mathbb{Z})/\mathrm{Tors}$ is isometric to $\langle -E_8 \rangle \oplus \langle H \rangle$, where H is the hyperbolic plane and E_8 stands for the unimodular, positive definite, even form of rank 8.

Every two Enriques surfaces are deformations of each other, therefore it make sense to talk about the oriented diffeomorphism (homeomorphism, homotopy) type of an Enriques surface.

A proof for all these facts can be found in [2].

Definition. *A homotopy Enriques surface is a surface which has the oriented homotopy type of an Enriques surface. A fake Enriques surface is a homotopy Enriques surface which is not an Enriques surface.*

Since $p_g(X)$ and $q(X)$ are invariants of the oriented homotopy type, it follows immediately from the Enriques–Kodaira classification that a fake Enriques surface must be a minimal, properly elliptic surface [2]. Thus every homotopy Enriques surface X admits an elliptic fibration $\pi: X \to \mathbb{P}^1$ over $\mathbb{P}^1$.

Since $\pi_1(X) = \mathbb{Z}/2$ is Abelian, this fibration can have at most two multiple fibres $\bar{p}F_{\bar{p}}$, $\bar{q}F_{\bar{q}}$ [4]. For such a surface one has

$$2 = |\pi_1(X)| = g.c.d.(\bar{p}, \bar{q}).$$

So we can write $\bar{p} = 2p$, $\bar{q} = 2q$ with relatively prime positive integers p, q [4]. From the canonical bundle formula for an elliptic surface we find [2]

$$K \sim F - F_{2p} - F_{2q}$$

for a canonical divisor of a homotopy Enriques surface. Here F denotes a generic fibre of π.

Lemma. *Let $\pi: X \to \mathbb{P}^1$ be an elliptic surface with $p_g(X) = q(X) = 0$ and multiple fibres $2pF_{2p}$, $2qF_{2q}$, where p, q are relatively prime positive integers. The following are equivalent:*

(i) *the universal covering* $\tilde{X}$ *of* X *is homeomorphic to a K3 surface*
(ii) *the intersection form on* $H^2(X;\mathbb{Z})_f$ *is even*
(iii) $p+q \equiv 0(2)$.

Proof. The universal covering $\tilde{X} \xrightarrow{\tau} X$ is the covering determined by the 2-torsion element $T \sim pF_{2p} - qF_{2q}$. The Steinfactorization of the composition $\pi \circ \tau$ defines a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tau} & X \\ {\scriptstyle \tilde{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{P}^1 & \xrightarrow{\tilde{\tau}} & \mathbb{P}^1 \end{array}$$

where $\tilde{\tau}: \mathbb{P}^1 \to \mathbb{P}^1$ is a double covering branched over $\pi(F_{2p})$ and $\pi(F_{2q})$.

In particular we have for a canonical divisor $\tilde{K}$ on $\tilde{X}$

$$\tilde{K} \sim (p-1)\tilde{F}_p + (q-1)\tilde{F}_q,$$

with $p\tilde{F}_p$, $q\tilde{F}_q$ the multiple fibres of $\tilde{\pi}$.

This description identifies $\tilde{X}$ as a homotopy $K3$ surface in the sense of Kodaira [20] iff $p+q \equiv 0(2)$. But by Freedman's classification theorem all homotopy $K3$ surfaces are in fact homeomorphic to a $K3$ surface. Therefore (i)$\Leftrightarrow$(iii).

To show (ii)$\Leftrightarrow$(iii) we remark that the intersection form on $H^2(X;\mathbb{Z})_f$ is even iff $D \cdot K \equiv 0(2)$ for every divisor D on X. This follows from the adjunction formula and the fact that every cohomology class in $H^2(X;\mathbb{Z})$ is the Chern class of a divisor. Using the formula for $\tilde{K}$ we find

$$2D\cdot K = (\tau^* D)\cdot(\tau^* K) = (\tau^* D)\cdot\tilde{K} = (p-1)\,(\tau^* D)\cdot\tilde{F}_p + (q-1)\,(\tau^* D)\cdot\tilde{F}_q.$$

Since $(\tau^* D)\cdot\tilde{F}_p$ and $(\tau^* D)\cdot\tilde{F}_q$ are even, $D\cdot K \equiv 0(2)$ iff p and q are both odd. This proves (ii)$\Leftrightarrow$(iii).

From now on $X_{2p,2q}$ will always denote an elliptic surface $\pi: X \to \mathbb{P}^1$ with $p_g(X) = q(X) = 0$ containing precisely two multiple fibres $2pF_{2p}$, $2qF_{2q}$ $p, q \geq 1$ relatively prime with $p+q \equiv 0(2)$.

We want to show that the oriented homotopy type of $X_{2p,2q}$ is independent of p and q. For this purpose we use a criterion of Hambleton and Kreck [15]. They prove that the oriented homotopy type of an oriented, closed topological 4-manifold X with finite fundamental group π_1, which has periodic cohomology of period 4, is determined by its quadratic 2-type. This quadratic 2-type is the isometry class of the quadruple $[\pi_1(X), \pi_2(X), S_{\tilde{X}}, k(X)]$, where $\pi_2(X)$ is considered as $\pi_1(X)$-module, $S_{\tilde{X}}$ is the intersection form of the universal covering $\tilde{X}$ of X and $k(X) \in H^3(\pi_1(X); \pi_2(X))$ is the first k-invariant of X [15]. In other words, the oriented homotopy type of such a manifold X is determined by its k-invariant and the equivariant intersection form of $\tilde{X}$.

$$\begin{array}{ccc} H^2(\tilde{X};\mathbb{Z}) \otimes H^2(\tilde{X};\mathbb{Z}) & \to & \mathbb{Z}[\pi_1(X)] \\ \alpha \otimes \beta & \longrightarrow & \sum\limits_{\sigma\in\pi_1(M)} (\alpha \cup \sigma^*\beta)\sigma^{-1} \end{array}$$

with $\sigma \in \pi_1(X)$ acting on $\tilde{X}$ as covering transformation. To apply this in our situation let

$$\sigma_{p,q}: \tilde{X}_{2p,2q} \to \tilde{X}_{2p,2q}$$

be the non-trivial covering involution. The standard involution ρ on the $K3$ lattice $2\langle -E_8\rangle \oplus 3\langle H\rangle$ is given by the formula [2]

$$\rho: 2\langle -E_8\rangle \oplus 3\langle H\rangle \to 2\langle -E_8\rangle \oplus 3\langle H\rangle$$

$$(x_1, x_2, y_1, y_2, y_3) \to (x_2, x_1, -y_1, y_3, y_2).$$

LEMMA. *There exists an isometry*

$$\varphi_{p,q}: H^2(\tilde{X}_{2p,2q}; \mathbb{Z}) \to 2\langle -E_8\rangle \oplus 3\langle H\rangle$$

such that

$$\varphi_{p,q} \circ \sigma^*_{p,q} \circ \varphi^{-1}_{p,q} = \rho.$$

Proof. This follows immediately from the uniqueness of the embedding of the Enriques lattice multiplied by 2 into the $K3$ lattice [26].

So far we have seen that the equivariant intersection forms of all surfaces $X_{2p,2q}$ are isometric. Now we consider their k-invariants. These invariants are elements in $H^3(\pi_1; \pi_2) \cong H^{-1}(\pi_1; \pi_2)$, where $\pi_1 = \mathbb{Z}/2$ and $\pi_2 = 2\langle -E_8\rangle \oplus 3\langle H\rangle$ has the π_1-module structure defined by ρ. Therefore $k(X_{2p,2q}) \in H/2H \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Since the k-invariants are always non-trivial, there are three possibilities: (0, 1), (1, 0), and (1, 1). The first two cases are equivalent under an isometry, so there exist at most two oriented homotopy types with the equivariant intersection form of an Enriques surface. In fact the example in [15] shows that there are two such types, namely $S(\eta \oplus \varepsilon^{\oplus 2}) \# |-E_8| \# S^2 \times S^2$ and $S(\eta^{\oplus 3}) \# |-E_8| \# S^2 \times S^2$. Here η and ε denote the canonical and the trivial line bundle over $\mathbb{P}^2_{\mathbb{R}}$, $S(\eta \oplus \varepsilon^{\oplus 2})$ and $S(\eta^{\oplus 3})$ their 2-sphere bundles; $|-E_8|$ is the closed, simply connected topological 4-manifold with intersection form $\langle -E_8\rangle$ [11]. The manifold $S(\eta \oplus \varepsilon^{\oplus 2}) \# |-E_8| \# S^2 \times S^2$ has k-invariant (1,0) and second Stiefel Whitney class $w_2 = 0$, the other manifold has k-invariant (1,1) and $w_2 \neq 0$.

Since by Rohlin's theorem [10] $S(\eta \oplus \varepsilon^{\oplus 2}) \# |-E_8| \# S^2 \times S^2$ does not admit a differentiable structure, the surfaces $X_{2p,2q}$ must all be oriented homotopy equivalent to $S(\eta^{\oplus 3}) \# |-E_8| \# S^2 \times S^2$.

THEOREM. *The homotopy Enriques surfaces are precisely the surfaces of type $X_{2p,2q}$ with $p, q \geqq 1$ relatively prime, odd integers.*

From the canonical bundle formula we obtain

$$K \sim F - F_{2p} - F_{2q}$$

for a canonical divisor K on $X_{2p,2q}$. In particular $2K \sim 0$ if and only if $p = q = 1$, so the fake Enriques surfaces are the $X_{2p,2q}$ with $\max(p,q) > 1$. It is easy to construct such surfaces:

Choose two generic homogeneous polynomials $Q_0(Y_0, Y_1, Y_2)$, $Q_1(Y_0, Y_1, Y_2)$ of degree 3 and consider the surface $X \subset \mathbb{P}^1 \times \mathbb{P}^2$ with equation $X_0 Q_1(Y) - X_1 Q_0(Y) = 0$, where (X_0, X_1) and (Y_0, Y_1, Y_2) are homogeneous coordinates on $\mathbb{P}^1$ and $\mathbb{P}^2$ respectively. We have two induced projection maps:

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & \mathbb{P}^2 \\ {\scriptstyle\pi}\downarrow & & \\ \mathbb{P}^1 & & \end{array}$$

$\sigma: X \to \mathbb{P}^2$ is the blow-up of the intersection of the two cubics $(Q_0)_0, (Q_1)_0 \subset \mathbb{P}^2$; $\pi: X \to \mathbb{P}^1$ defines on X an elliptic structure with $\pi^{-1}[0, 1] \cong (Q_0)_0$, $\pi^{-1}[1, 0] \cong (Q_1)_0$. Now we choose points of order $2p$ and $2q$ in the Jacobians of the fibres over [0, 1] and [1, 0] respectively and perform the corresponding logarithmic transformations [28]. The result is a surface $X_{2p,2q}$. For a generic choice of Q_0 and Q_1 this surface will have the two multiple fibres and 12 singular fibres of type I_1. In particular there will be no reducible fibre.

An explicit example can be obtained as follows: Consider a Halphen pencil $\lambda F_{6q} + (1-\lambda)F_3^{2q}$ of plane curves with nine $2q$-multiple points [28]. Let X be a minimal non-singular model of the cyclic plane $Z^{2p} = F_{6q}$. The inverse image of the Halphen pencil defines an elliptic pencil on X. Since this pencil contains two multiple fibres, the inverse image of the curve $(F_3^{2q})_0$ and the ramification curve, X is a surface of type $X_{2p,2q}$.

This example has been communicated by the referee.

2. TOPOLOGY OF FAKE ENRIQUES SURFACES

In this section we use topological surgery to show that every fake Enriques surface is homeomorphic to an "ordinary" Enriques surface.

Let X be any closed, oriented topological 4-manifold with fundamental group π_1. A topological smoothing of X is an equivalence class of pairs (h, M), M an oriented topological 4-manifold, $h: M \to X$ an orientation preserving simple homotopy equivalence. Another pair (h', M') is equivalent to (h, M) if there exists an orientation preserving homeomorphism $\alpha: M \to M'$ such that $h' \circ \alpha \simeq h$ [31]. Let $\mathscr{S}_{\mathrm{top}}(X)$ denote the set of all topological smoothings of X.

This set has a natural action of the group $\mathrm{Aut}(X)$ of homotopy classes of simple self equivalences; the orbit space $\mathscr{S}_{\mathrm{top}}(X)/\mathrm{Aut}(X)$ under this action parametrizes homeomorphism classes of topological manifolds in the oriented simple homotopy type of X. For "good" fundamental groups π_1 [12] one has the following exact surgery sequence:

$$(*) \qquad L_5^s(\pi_1^+) \to \mathscr{S}_{\mathrm{top}}^s(X) \xrightarrow{\nu} \mathscr{N}_{\mathrm{top}}(X) \xrightarrow{\vartheta} L_4^s(\pi_1^+).$$

Here $L_m^s(\pi_1^+)$ is a shorthand notation for the mth Wall group $L_m^s(\pi_1, w)$, with $w: \pi_1 \to \{\pm 1\}$ the trivial orientation homomorphism [31]. $\mathscr{S}_{\mathrm{top}}^s(X)$ denotes the set of s-cobordism classes of orientation preserving simple homotopy equivalences [18], $\mathscr{N}_{\mathrm{top}}(X)$ is the set of normal invariants of X. The elements of $\mathscr{N}_{\mathrm{top}}(X)$ are bordism classes of normal maps (f, b)

$$\begin{array}{ccc} \nu_M & \xrightarrow{b} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X, \end{array}$$

where ν_M is the stable topological normal bundle of M, b a stable bundle map covering the degree 1 map f [18].

The surgery obstruction ϑ associates to a normal map (f, b) the obstruction to constructing a normal bordism to a simple homotopy equivalence [12].

According to Freedman [12], if this obstruction vanishes, such a bordism can be constructed. This proves exactness of $(*)$ at $\mathscr{N}_{\mathrm{top}}(X)$. Exactness at $\mathscr{S}_{\mathrm{top}}^s(X)$ is known from [31] in the following stronger sense: $L_5^s(\pi_1^+)$ acts on $\mathscr{S}_{\mathrm{top}}^s(X)$ such that two elements are in the same orbit precisely if their ν-images are equal.

Now Freedman's topological s-cobordism theorem for good fundamental groups allows us to identify $\mathscr{S}_{\mathrm{top}}^s(X)$ with $\mathscr{S}_{\mathrm{top}}(X)$ [12].

Finally the set of normal invariants $\mathcal{N}_{\text{top}}(X)$ has a homotopy-theoretic interpretation; $\mathcal{N}_{\text{top}}(X)$ is in 1-1-correspondence with the group of homotopy classes of maps from X to the H-space G/Top, the fibre of the fibration $B\text{Top} \to BG$. Here $B\text{Top}$ and BG are classifying spaces for stable topological bundles and stable fibre homotopy equivalences respectively [18].

The first non-trivial homotopy groups of G/Top are $\pi_2(G/\text{Top}) = \mathbb{Z}/2$, $\pi_4(G/\text{Top}) = \mathbb{Z}$ with vanishing k-invariant in $H^5(K(\mathbb{Z}/2, 2); \mathbb{Z})$. Therefore the Postnikov resolution of G/Top gives an H-map

$$G/\text{Top} \to K(\mathbb{Z}/2, 2) \times K(\mathbb{Z}, 4),$$

which is a 5-equivalence [18]. In particular we have

$$\mathcal{N}_{\text{top}}(X) = [X, G/\text{Top}] = [X, K(\mathbb{Z}/2, 2) \times K(\mathbb{Z}, 4)] = H^2(X; \mathbb{Z}/2) \oplus H^4(X; \mathbb{Z}).$$

Putting all this together we can rewrite the surgery sequence in the following form:

$$(*) \qquad L_5^s(\pi_1^+) \to \mathcal{S}_{\text{top}}(X) \xrightarrow{\nu} H^2(X; \mathbb{Z}/2) \oplus H^4(X; \mathbb{Z}) \xrightarrow{\vartheta} L_4^s(\pi_1^+).$$

We apply this to an Enriques surface X.

Lemma. *Let X be the oriented, topological manifold underlying an Enriques surface. The normal invariant ν defines a bijection*

$$\nu : \mathcal{S}_{\text{top}}(X) \to H^2(X; \mathbb{Z}/2).$$

Proof. For any finite cyclic group π_1 the Wall group $L_5^s(\pi_1^+)$ vanishes, so ν identifies $\mathcal{S}_{\text{top}}(X)$ with the inverse image $\vartheta^{-1}(0)$ of 0 of the surgery obstruction ϑ. Since in dimension 4 ϑ is a homomorphism, injective on the summand $H^4(X; \mathbb{Z})$ [31], the claim follows.

To determine the set of topological manifolds in the oriented simple homotopy type of X we have to divide by the group $\text{Aut}(X)$ of simple self equivalences. Since for $\pi_1 = \mathbb{Z}/2$ the Whitehead group $\text{Wh}(\pi_1)$ vanishes [31], every homotopy equivalence is simple. Let $f: X \to X$ be a self equivalence, $h: M \to X$ a representative of a topological smoothing. The action of $[f]$ on $[h, M]$ is given by $[h, M] \to [f \circ h, M]$. Applying ν we get

$$\nu([f] \cdot [h, M]) = \nu([f, X]) + (f^{-1})^* \nu([h, M]).$$

Now let

$$\Delta : H^2(X; \mathbb{Z}/2) \to H_2(X; \mathbb{Z}/2)$$

denote Poincaré duality; then

$$\Delta\nu([f] \cdot [h, M]) = \Delta\nu([f, X]) + f_* \Delta\nu([h, M]).$$

Consider the universal covering $\tau : \tilde{X} \to X$. Wall shows [31] that for every element α in the image of

$$\tau_* : H_2(\tilde{X}; \mathbb{Z}/2) \to H_2(X; \mathbb{Z}/2)$$

there exists an $f_\alpha \in \text{Aut}(X)$ with $\Delta\nu([f_\alpha, X]) = \alpha$ inducing the identity map $f_{\alpha_*} = \text{id}$ on $H_2(X; \mathbb{Z}/2)$. In other words, the orbit space $\mathcal{S}_{\text{top}}(X)/\text{Aut}(X)$ contains at most as many elements as

$$\text{Cok } \tau_* = H_2(\pi_1; \mathbb{Z}/2).$$

Since $\pi_1(X) = \mathbb{Z}/2$ we have $|\text{Cok } \tau_*| = 2$. So *a priori* there could be two different oriented, topological 4-manifolds homotopy equivalent to an Enriques surface X. But fortunately we know a little more: X is homotopy equivalent to a connected sum

$$X \simeq S(\eta^{\oplus 3}) \# |-E_8| \# S^2 \times S^2.$$

In this situation there exists only one homeomorphism type [21]. This shows:

THEOREM. *Every two homotopy Enriques surfaces are homeomorphic.*

3. DIFFERENTIABLE STRUCTURES OF FAKE ENRIQUES SURFACES

At the end of section 1 we constructed for every pair of relatively prime, odd, positive integers (p, q) generic homotopy Enriques surfaces $X_{2p,2q}$. We did this by applying logarithmic transformation of multiplicities $2p$ and $2q$ at two non-singular fibres of a rational elliptic surface X, the blow-up of the nine basepoint of a generic pencil of cubics in $\mathbb{P}^2$. This construction obviously involves a number of choices; but the diffeomorphism type of the surfaces $X_{2p,2q}$ depends only on the set $\{p, q\}$ [16].

In this paragraph we will show that the map which associates to such a set $\{p, q\}$ the diffeomorphism type of $X_{2p,2q}$ is finite-to-one.

To achieve this we have to compute the Donaldson invariant of a generic homotopy Enriques surface. Let us recall the definition and the relevant properties of this invariant. Consider a closed, oriented smooth 4-manifold X with finite first homology group $H_1(X; \mathbb{Z})$ and intersection form of type $(1, n)$ on $H^2(X; \mathbb{R})$. Then the positive cone

$$\Omega = \{\alpha \in H^2(X; \mathbb{R}) | \ \alpha^2 > 0\}$$

has two connected components interchanged by multiplication with -1. Each integral class $e \in H^2(X; \mathbb{Z})_f$ with $e^2 = -1$ defines a wall

$$W_e = \{\alpha \in \Omega | \alpha \cdot e = 0\}$$

in Ω. Their union $\bigcup_{e^2=-1} W_e$ decomposes Ω into chambers, the connected components of $\Omega \setminus \bigcup_{e^2=-1} W_e$. The Donaldson invariant is a map

$$\Gamma_X : \mathscr{C}_X \to H^2(X; \mathbb{Z})$$

on the set $\mathscr{C}_X$ of these chambers with the following formal properties:

(P1) $\Gamma_X(-C) = -\Gamma_X(C)$

(P2) If C_+, C_- are chambers in the same component of Ω such that there is precisely one wall W_e with $C_- \cdot e < 0 < C_+ \cdot e$ then

$$\Gamma_X(C_+) = \Gamma_X(C_-) + 2\sum (e + t)$$

where the sum is taken over all torsion classes in $H^2(X; \mathbb{Z})$

(P3) If $f: X' \to X$ is an orientation preserving diffeomorphism then

$$\Gamma_{X'}(f^* C) = f^* \Gamma_X(C).$$

Thus Γ_X is completely determined by its value on any chamber.

For a specific chamber C_0, $\Gamma_X(C_0)$ will be defined by the homology classes of certain Yang–Mills moduli spaces.

Let P be the principal SU(2)-bundle on X with $c_2(P) = 1$. Denote by $\mathscr{B}$ the infinite dimensional space of gauge equivalence classes of connections on P and by $\mathscr{B}^*$ the open subspace of irreducible connections. Each Riemannian metric g on X defines a Hodge operator $*_g$. Since by assumption the intersection form is of type $(1, n)$ on $H^2(X; \mathbb{R})$, there exists a unique line $H^2_+ \subset H^2(X; \mathbb{R})$ represented by $*_g$-self-dual harmonic 2-forms.

The metric g also defines the finite dimensional subspace $M(g) \subset \mathscr{B}$ of connections with $*_g$-anti-self-dual curvature,

$$M(g) = \{[A] \in \mathscr{B} \mid *_g F_A = -F_A\}.$$

This space contains reducible connection precisely when H^2_+ hits one of the walls W_e. For each such wall $W_e = W_{-e}$ there are $|H_1(X; \mathbb{Z})|$ reducible points in $M(g)$.

For a generic metric g, $M(g)$ will be a smooth 2-dimensional manifold not containing any reducible connection. It can be oriented; an orientation corresponds to the choice of a form $\omega_g \in H^2_+$ of norm 1. In general $M(g)$ has several ends; the Taubes' construction defines finite dimensional models for these ends and provides a method to compactify $M(g)$. Now on $\mathscr{B}^* \times X$ there exists a universal $U(2)$-bundle $\mathbb{P}$. Its second Chern class defines a map

$$\hat{\mu}: H_2(\mathscr{B}^*; \mathbb{Z}) \to H^2(X; \mathbb{Z})$$

$$m \to m \backslash c_2(\mathbb{P}).$$

If $M(g)$ happened to be compact the choice of a normalized form $\omega_g \in H^2_+$ would define a fundamental class $[M(g)] \in H_2(\mathscr{B}^*; \mathbb{Z})$ and a corresponding element $[M(g)] \backslash c_2(\mathbb{P}) \in H^2(X; \mathbb{Z})$

Since in general $M(g)$ is not compact Donaldson uses a "correction term" corresponding to the ends of $M(g)$ to associate an element $\Gamma(C_0) \in H^2(X; \mathbb{Z})$ to the chamber C_0 containing ω_g.

If ω_g considered as a section in the bundle Λ^2_+ of self-dual 2-forms has no zeros and if $M(g)$ is compact this cohomology class is

$$\Gamma_X(C_0) = aK + 2\hat{\mu}([M(g)]).$$

Here a is the number of 2-torsion elements in $H_1(X; \mathbb{Z})$ and K denotes the Euler class of the quotient bundle $\Lambda^2_+/\mathbb{R}\omega_g$.

We will use this formula for the Kähler form ω_g of a Hodge metric g on a complex projective surface. In this case ω_g has no zeros and the quotient $\Lambda^2_+/\mathbb{R}\omega_g$ can be identified with the canonical line bundle of X [9]; so K is the class of a canonical divisor. For an algebraic surface with ample line bundle H and corresponding metric g the space

$$M(g) \cap \mathscr{B}^*$$

of irreducible anti-dual connections has a complex structure [22]. Using Donaldson's solution of the Kobayashi–Hitchin conjecture [5], it can be identified with the moduli space $M(H)$ of H-stable [24] rank-2 vector bundles E with trivial determinant and second Chern class $c_2(E) = 1$.

After these general preliminaries we return to our special situation.

LEMMA. *Let X be an algebraic surface with fundamental group $\pi_1 = \mathbb{Z}/2$, H an ample line bundle on X, g the associated metric. If X has an even intersection form then the moduli space $M(g)$ is contained in $\mathscr{B}^*$ and the corresponding space $M(H)$ is a complete projective variety.*

Proof. Of course, $M(g)$ is contained in $\mathscr{B}^*$ for every metric g since there are no (-1)-classes in $H^2(X; \mathbb{Z})$. To prove that $M(H)$ is a projective variety we consider the natural Gieseker-compactification $\overline{M(H)}$ [14] and show $\overline{M(H)} \backslash M(H) = \phi$.

The points in $\overline{M(H)}$ are S-equivalence classes of torsion free rank-2 sheaves E with trivial determinant and second Chern class $c_2(E) = 1$, which are semi-stable in the sense of Gieseker [14]. This means that every torsion free rank-1 quotient Q of E has either $c_1(Q)\cdot H > 0$ or $c_1(Q)\cdot H = 0$ and $c_1(Q)^2 - c_1(Q)\cdot K - 2c_2(Q) \geqq -1$ [14]. Now every such quotient is of the form $I_Z L^\vee$, $Z \subset X$ a subvariety of finite length $|Z|$, L a line bundle. Therefore points in $\overline{M(H)}\setminus M(H)$ are given by torsion free sheaves E, which either are H-stable but not locally free or admit a quotient $E \twoheadrightarrow I_Z L^\vee$ with $L\cdot H = 0$, $\chi(L^\vee) - \chi(\mathcal{O}_X) \geqq |Z|$. The first possibility is excluded since the double dual $E^{\vee\vee}$ of E would be an H-stable vector bundle with $c_1 = c_2 = 0$. In the second case we would have an exact sequence

$$0 \to I_W L \to E \to I_Z L^\vee \to 0,$$

$$0 \leqq |W| + |Z| = 1 + L^2, \quad L\cdot H = 0, \quad \chi(L^\vee) - \chi(\mathcal{O}_X) \geqq |Z|.$$

The Hodge index theorem implies $L^2 \leqq 0$, so $L^2 = 0$ if the intersection form is even.

Then L is a torsion element, $Z = \phi$, W a simple point and the sequence

$$0 \to I_W L \to E \to L^\vee \to 0$$

must not split. But if $\pi_1 = \mathbb{Z}/2$, L is a 2-torsion element and $\mathrm{Ext}^1(L^\vee, I_W L) = 0$. This shows that $M(H)$ is complete for every polarization H.

Now let $X_{2p,2q}$ be a generic homotopy Enriques surface with multiple fibres $2pF_{2p}$, $2qF_{2q}$. We choose an ample divisor $\hat{H}_{p,q}$ on every $X_{2p,2q}$ and define

$$H_{p,q} = \hat{H}_{p,q} + m_{p,q} K_{p,q},$$

where $K_{p,q} \sim -F + (2p-1)F_{2p} + (2q-1)F_{2q}$ is a canonical divisor, $m_{p,q} \gg \hat{H}_{p,q}\cdot K_{p,q}$ a sufficiently large integer. Then $H_{p,q}$ is also ample. Let $M(H_{p,q})$ be the moduli space of $H_{p,q}$-stable rank-2 bundles E with $c_1(E) = 0$ and $c_2(E) = 1$ on $X_{2p,2q}$.

Proposition. *For a generic homotopy Enriques surface $X_{2p,2q}$ the moduli space $M(H_{p,q})$ is a disjoint union of complete curves. The normalization of the reduction of a component is isomorphic to F_{2p} or to F_{2q}. There exists a universal bundle $\mathbb{E}$ on $M(H_{p,q}) \times X$ whose restriction to F_{2s} $(s = p,q)$ is given by the universal extension*

$$0 \to \pi_2^* \mathcal{O}_{X_{2p,2q}}(D - K_{p,q}) \to \mathbb{E}|_{F_{2s} \times X_{2p,2q}} \to I_\Delta \otimes \pi_2^* \mathcal{O}_{X_{2p,2q}}(K_{p,q} - D) \otimes \pi_1^* \mathcal{O}_{F_{2s}}(2D - 3K_{p,q}) \to 0$$

Here π_i projects $F_{2s} \times X_{2p,2q}$ onto the ith factor, $D \subset X_{2p,2q}$ is a curve depending on the component and $\Delta \subset F_{2s} \times X_{2p,2q}$ is the graph of the imbedding $F_{2s} \subset X_{2p,2q}$.

Proof. The proof of this proposition is completely analogous to the corresponding proof in [27]; so we sketch the proof for the special case $p = 1$ and leave the general case to the reader. Let $X = X_{2,2q}$, $H = H_{1,q}$, $K = K_{1,q}$. Using Riemann–Roch and the definition of stability one finds that every bundle $E \in M(H)$ is given by an extension

$$0 \to \mathcal{O}_X(D - K) \to E \to I_z(K - D) \to 0,$$

$D \subset X$ a vertical curve, $z \in X$ a simple point in special position with respect to the linear system $|3K - 2D|$. From $(D - K)H < 0$ one obtains

$$D \sim dF_{2q}, \quad 0 \leqq d \leqq q - 2.$$

So for each E as above there is a unique maximal curve $D_E \sim d_E F_{2q}$, $0 \leqq d_E \leqq q - 2$, such that $E(K - D_E)$ has a section vanishing simply in a point $z \in X$. Since $H^\circ(E(K - D_E))$ is 1-dimensional, this point $z = z_E$ is uniquely determined by E. It is in special position with respect to $|3K - 2D_E|$ precisely if it is contained in F_2. Conversely, given any d with

$0 \leqq d \leqq q-2$ and a point $z \in F_2$ there exists a unique H-stable 2-bundle E with $d_E = d$, $z_E = z$.

Now fix d with $0 \leqq d \leqq q-2$ and let $\Delta \subset F_2 \times X$ denote the graph of the imbedding $F_2 \subset X$. On $F_2 \times X$ one can construct a universal extension [1]

$$0 \to \pi_2^* \mathcal{O}_X(dF_{2q} - K) \to \mathbb{E}_d \to I_\Delta \otimes \pi_2^* \mathcal{O}_X(K - dF_{2q}) \otimes \pi_1^* \mathcal{O}_{F_2}(2dF_{2q} - 3K) \to 0,$$

which restricts over $\{z\} \times X$ to

$$0 \to \mathcal{O}_X(dF_{2q} - K) \to E \to I_z(K - dF_{2q}) \to 0,$$

$d = d_E, \quad z = z_E$ [27].

On the other hand we know from Maruyama's criterion [24] that $M(H)$ is a fine moduli space. Using the universal property we obtain a map

$$\varepsilon\colon \coprod_{d=0}^{q-2} F_2 \times \{d\} \to M(H)$$

which classifies $\coprod_{d=0}^{q-2} \mathbb{E}_d$ and is a bijection. Therefore the normalization of the reduction of $M(H)$ must be isomorphic to the disjoint union of $q-1$ copies of F_2.
This proves the proposition, at least for $p=1$, $q \geqq 1$.

There is one difference between the special case $p = 1$ and the general situation. If $p = 1$ then every bundle E in $M(H_{1,q})$ has $h^2(\mathrm{End}\, E) = 0$, which means that the moduli space is smooth at every point; thus

$$M(H_{1,q}) = \coprod_{d=0}^{q-2} F_2 \times \{d\}.$$

In general the components of $M(H_{p,q})$ will have non-reduced structures.

We can now compute the Donaldson invariants for generic homotopy Enriques surfaces $X_{2p,2q}$. Since $X_{2p,2q}$ has an even intersection form, we have only two chambers, the components Ω_+, Ω_- of the positive cone. We can assume that Ω_+ contains the Kähler form corresponding to $H_{p,q}$. Let $T_{p,q} = pF_{2p} - qF_{2q}$. We choose integers a, b with $ap + bq = 1$ and define

$$\hat{\Gamma}_{p,q} = bF_{2p} + aF_{2q}.$$

The image $\Gamma_{p,q}$ of $\hat{\Gamma}_{p,q}$ in $H^2(X_{2p,2q}; \mathbb{Z})_f$ is a primitive class, independent of the choice of a, b.

The preceding proposition gives

Corollary. *Let Ω_+ denote the component of Ω containing the Kähler class corresponding to $H_{p,q}$. Then*

$$\Gamma_{X_{2p,2q}}(\Omega_+) = n_{p,q} \Gamma_{p,q},$$

where $n_{p,q}$ is a natural number with

$$n_{p,q} \geqq 2(2pq - (p+q)).$$

Proof. $\Gamma_{X_{2p,2q}}(\Omega_+) = 2K_{p,q} + 2\hat{\mu}([M(H_{p,q})])$. Since

$$K_{p,q} \sim (2pq - (p+q))\hat{\Gamma}_{p,q} - (a+b)T_{p,q},$$

all we have to show is

$$\hat{\mu}([M(H_{p,q})]) = r\hat{\Gamma}_{p,q} + sT_{p,q}$$

for integers r, s with $r \geqq 0$. From the universal extension we see that $\hat{\mu}([M(H_{p,q})])$ is the Poincaré dual of a non-negative divisor of the form $r_p F_{2p} + r_q F_{2q}$. But $F_{2p} \sim q\hat{\Gamma}_{p,q} + a\, T_{p,q}$, $F_{2p} \sim p\hat{\Gamma}_{p,q} + b\, T_{p,q}$, which proves the corollary.

If $p = 1$ we know the precise value of $n_{1,q}$:

$$\Gamma_{X_{2,2q}}(\Omega_+) = (q+2)\,(q-1)F_{2q}.$$

These numbers $n_{p,q}$ – the orders of divisibility of $\Gamma_{X_{2p,2q}}(\Omega_+)$ – are invariants of the diffeomorphism type of $X_{2p,2q}$. In fact, consider a diffeomorphism

$$f\colon X_{2p,\,2q} \to X_{2p',\,2q'};$$

f is automatically orientation preserving. Let Ω_+ and Ω'_+ contain the Kähler forms corresponding to $H_{p,q}$ and $H_{p',q'}$ respectively; then $f^*\Omega'_+ = \pm\Omega_+$. So using the formal properties of the Donaldson invariant we obtain

$$n_{p',\,q'} f^*\Gamma_{p',\,q'} = f^*\Gamma_{X_{2p',2q'}}(\Omega'_+) = \Gamma_{X_{2p,2q}}(f^*\Omega'_+) = \pm\Gamma_{X_{2p,2q}}(\Omega_+) = \pm n_{p,q}\Gamma_{p,q},$$

which implies $n_{p',q'} = n_{p,q}$. Since the $n_{p,q}$ are bounded from below by $2(2pq-(p+q))$, we have

THEOREM. *The map, which associates to a pair (p, q) of relative prime, odd, positive integers the diffeomorphism type of a homotopy Enriques surface $X_{2p,\,2q}$ is finite-to-one.*

Knowing the precise values $\Gamma_{X_{2,2q}}(\Omega_+) = (q+2)\,(q-1)F_{2q}$ for surfaces of type $X_{2,\,2q}$ we see

COROLLARY. *Homotopy Enriques surfaces of type $X_{2,2q}$, $X_{2,\,2q'}$ are diffeomorphic if and only if $q = q'$.*

The Donaldson invariant for an ordinary Enriques surface $X_{2,2}$ vanishes. On the other hand $2(2pq-(p+q)) > 0$ unless $p = q = 1$, therefore

COROLLARY. *No fake Enriques surface can be diffeomorphic to an ordinary Enriques surface.*

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